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# The Green function for the step potential via an exact summation of the perturbation series

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**Abstract**
The path integral treatment of the step potential based on exact summation of the Feynman perturbation series is presented in its original form. With a two-sided Laplace transform on the end point, we have established a recurrence formula between three successive terms of the perturbation series which leads us to sum up exactly all the terms and deduce the Green function of the problem. The Wiener–Hopf method and Hilbert problems are successfully used for solving our problem with the boundary conditions.

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## 1. Introduction

As we know, most physical problems cannot be treated exactly. Then, it is necessary to develop some approximation procedures which lead us to approach the exact result with an appropriate accuracy. The most important and usual approximation procedure method for solving problems in quantum mechanics is perturbation theory in the Schroëdinger formalism. It provides us with an effective method to compute the approximate solutions of many problems which cannot be exactly solved by using the Schroëdinger equation. As in standard quantum mechanics, the perturbation method can be developed in the path integral framework of quantum mechanics [1].

Since 1970, perturbation expansion of the path integral has been used to give the exact Green functions for the delta-function potential problem [3, 4], non-relativistic Coulomb system [2, 5], and to yield the boundary conditions for the non-relativistic problems [6]. Also the perturbation approach was successfully used for deriving the energy Green function for the inverse square potential [7].

In this paper, we would like to add a further application and a contribution of the perturbation method to the path integral. This contribution, not treated to our knowledge and in its original form, concerns the energy Green function of a step potential via summation

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of the perturbation series. At this stage we mention that [8, 9] have derived Green function and the propagator for the step potential by solving the Schroëdinger equation directly; [10] also derived Green function of the same potential but by using the $\delta$ perturbation method on the Wood–Saxon potential and passing in the limit R $\to$ 0 and also [11] using the combinatorial approach derived Green function of the problem but none of these papers found in the literature treats the perturbation series of the step potential in its original framework as formulated by Feynman [1] earlier.

## 2. Path integral for the step potential via summation of the perturbation series

Consider the classical Lagrangian

$$L(x, \dot{x}, t) = \tfrac{1}{2}m\dot{x}^2 - V(x). \tag{1}$$

The Feynman propagator [1] is defined by

$$K(x, T/y, 0) = \int_{x(0)=y}^{x(T)=x} D[x(t)] \exp\left(\mathrm{i}\int_0^T L(x, \dot{x}, t)\,\mathrm{d}t\right), \tag{2}$$

where $D[x(t)]$ is the formal measure on the path space. If we split the Lagrangian into the free part and the interaction part, we can show that the Feynman propagator takes the form

$$K(x, T/y, 0) = \sum_{n=0}^{\infty} K_n(x, T/y, 0), \tag{3}$$

where

$$\begin{aligned} K_n(x, T/y, 0) = (-\mathrm{i})^n &\int_0^T \mathrm{d}t_n \cdots \int_0^{t_2} \mathrm{d}t_1 \\ &* \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=0}^{j=n} K_0(x_{j+1}, t_{j+1}/x_j, t_j) \prod_{j=1}^{j=n} V(x_j)\,\mathrm{d}x_j \end{aligned} \tag{4}$$

and $K_0(x_{j+1}, t_{j+1}/x_j, t_j)$ is the free particle propagator given by

$$K_0(x_{j+1}, t_{j+1}/x_j, t_j) = \left(\frac{m}{2\mathrm{i}\pi(t_{j+1} - t_j)}\right)^{1/2} \exp(\mathrm{i}m(x_{j+1} - x_j)^2/2(t_{j+1} - t_j)). \tag{5}$$

Now we take the Fourier transform of $K_n(x, T/y, 0)$ on $T$ as

$$g_n(x, y) = \mathrm{i}\int_0^{\infty} K_n(x, T/y, 0) \exp(-\mathrm{i}ET)\,\mathrm{d}T \tag{6}$$

which can be rewritten as

$$\begin{aligned} g_n(x, y) &= (-1)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=0}^{j=n} g_0(x_{j+1}, x_j) \prod_{j=1}^{j=n} V(x_j)\,\mathrm{d}x_j \\ &= (-1)^n \int_{-\infty}^{+\infty} \mathrm{d}x_n g_0(x, x_n) V(x_n) \prod_{j=0}^{j=n-1} g_0(x_{j+1}, x_j) \prod_{j=1}^{j=n-1} V(x_j)\,\mathrm{d}x_j \\ &= -\int_{-\infty}^{+\infty} \mathrm{d}x_n g_0(x, x_n) V(x_n) g_{n-1}(x_n, y), \end{aligned} \tag{7}$$

where

$$g_0(x, x_n) = \mathrm{i}\int_0^{\infty} \mathrm{d}T\, K_0(x, T : x_n, 0) \exp(-\mathrm{i}ET) = \frac{m}{k} \exp(-k|x - x_n|) \tag{8}$$

$k^2 = 2mE$ and

$$V(x) = V_0\theta(x), \qquad (V_0 > 0). \tag{9}$$

After substituting the last expression of the potential $V(x)$ in (7), we obtain

$$g_n(x, y) = -\frac{mV_0}{k}\int_0^{+\infty} \mathrm{d}\xi \, \exp(-k|x-\xi|) g_{n-1}(\xi, y). \tag{10}$$

In the following, we follow the notation used in [12] and construct the scaled functions as

$$g_n^+(x, y) = \begin{cases} 0; & x > 0 \\ g_n(x, y); & x < 0 \end{cases} \quad \text{and} \quad g_n^-(x) = \begin{cases} -g_n(x, y); & x > 0 \\ 0; & x < 0. \end{cases} \tag{11}$$

Using these notation, (10) transforms to

$$g_n^+(x, y) - g_n^-(x, y) = \frac{mV_0}{k}\int_{-\infty}^{+\infty} \mathrm{d}\xi \, \exp(-k|x-\xi|) g_{n-1}^-(\xi, y). \tag{12}$$

Let us now apply the two-sided Laplace transform on the end point $x$ in the last 'integral equation':

$$g_n^+(s, y) - g_n^-(s, y) = \frac{mV_0}{k} K(s) g_{n-1}^-(s, y), \tag{13}$$

where

$$g_n^+(s, y) = \int_{-\infty}^0 g_n(x, y) \exp(-sx) \, \mathrm{d}x; \qquad s < 0 \tag{14}$$

$$g_n^-(s, y) = -\int_0^{\infty} g_n(x, y) \exp(-sx) \, \mathrm{d}x; \qquad s > 0 \tag{15}$$

$$K(s) = \int_{-\infty}^{\infty} \mathrm{d}x \, \exp(-sx - k|x|) = \frac{2k}{k^2 - s^2} \qquad \text{with} \quad -k < s < k. \tag{16}$$

We point out here the fact that the perturbation method with the Laplace transformation technique was also developed by [2, 3] for a Coulomb and $\delta$ problem, respectively. Then we have to solve the recursive equation with three terms of the perturbation series:

$$g_n^+(s, y) - g_n^-(s, y) = \frac{-2mV_0}{s^2 - k^2} g_{n-1}^-(s, y). \tag{17}$$

We shall now start to solve this equation both in the regions $y < 0$ and $y > 0$. To accomplish this task, we use the non-homogeneous Hilbert problem stated in section 58 in [12].

## 3. Region $y < 0$

As in the case of the differential equations, we must use the initial conditions to generate the solution. Here we are in need of $g_0^-(s, y)$ and $g_0^+(s, y)$:

$$g_0^-(s, y) = \left(-\frac{m}{k}\right)\int_0^{\infty} \mathrm{d}x \, \exp(-sx - k|x-y|) = \left(-\frac{m}{k}\right)\frac{\exp(ky)}{s+k} \tag{18}$$

$$\begin{aligned} g_0^+(s, y) &= \left(\frac{m}{k}\right)\int_{-\infty}^{0} \mathrm{d}x \, \exp(-sx - k|x-y|) \\ &= \left(-\frac{m}{k}\right)\left(\frac{\exp(ky)}{s+k} + \frac{\exp(-sy)}{s-k} - \frac{\exp(-sy)}{s+k}\right) \end{aligned} \tag{19}$$

$$g_1^+(s,y) - g_1^-(s,y) = \left(\frac{m}{k}\right)\left(\frac{2mV_0}{s^2-k^2}\right)\left(\frac{\exp(ky)}{s+k}\right) \equiv f_1(s,y). \tag{20}$$

This corresponds to the non-homogeneous Hilbert problem which states that the solution of the last equation (formula 385 of the book [12]) takes the form

$$g_1^+(s,y) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau}{\tau-s}f_1(\tau,y) = -\text{residue}\left(\frac{f_1(\tau,y)}{\tau-s}\right)_{\tau=k} + (P(s)=0) \tag{21}$$

$$g_1^+(s,y) = \left(\frac{m}{k}\right)\frac{1}{4}\left(\frac{V_0}{E}\right)\frac{\exp(ky)}{s-k} \equiv C_1\left(\frac{-V_0}{E}\right)\left(\frac{m}{k}\right)\frac{\exp(ky)}{s-k}, \tag{22}$$

whereas $g_1^-(s,y)$ must be calculated from

$$g_1^-(s,y) = g_1^+(s,y) - f_1(s,y) = \left(\frac{m}{k}\right)\frac{1}{4}\left(\frac{V_0}{E}\right)\frac{\exp(ky)}{s-k}\left(1-\frac{(2k)^2}{(s+k)^2}\right). \tag{23}$$

Let us compute now $g_2^-(s,y)$ and $g_2^+(s,y)$ obeying the recurrence formula

$$g_2^+(s,y) - g_2^-(s,y) = \frac{-2mV_0}{s^2-k^2}g_1^-(s,y) \equiv f_2(s,y). \tag{24}$$

Using the same formula (385) of [12], we also deduce

$$\begin{aligned} g_2^+(s,y) &= \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}\tau}{\tau-s}f_2(\tau,y) \\ &= -\frac{1}{8}\left(\frac{V_0}{E}\right)^2\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right) \equiv C_2\left(-\frac{V_0}{E}\right)^2\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right) \end{aligned} \tag{25}$$

$$\begin{aligned} g_2^-(s,y) &= g_2^+(s,y) - f_2(s,y) \\ g_2^-(s,y) &= \left(-\frac{m}{k}\right)\frac{1}{8}\left(\frac{V_0}{E}\right)^2\left(1-\frac{1}{2}\frac{(2k)^2}{s^2-k^2}+\frac{(2k)^4}{2(s-k)(s+k)^3}\right) \end{aligned} \tag{26}$$

and it is easy to show in the same manner that

$$g_3^+(s,y) = \frac{5}{64}\left(\frac{V_0}{E}\right)^3\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right) \equiv C_3\left(-\frac{V_0}{E}\right)^3\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right). \tag{27}$$

If we follow the calculation of the successive terms $g_n^+(s,y)$, we easily remark that the coefficients $C_n$ of the factor $\left(-\frac{V_0}{E}\right)^n\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right)$ in each term $g_n^+(s,y)$ are nothing that

$$C_n = -\frac{1}{n!}\frac{\partial^n}{\partial u^n}\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)_{u=0} \tag{28}$$

which can be easily shown by recurrence. Then we write

$$g_n^+(s,y) = C_n\left(-\frac{V_0}{E}\right)^n\left(\frac{m}{k}\right)\left(\frac{\exp(ky)}{s-k}\right), \qquad \text{for} \qquad n \geqslant 1 \tag{29}$$

$$\begin{aligned} g^+(s,y) &= g_0^+(s,y) + \sum_{n=1}^{\infty} g_n^+(s,y) \\ &= -\frac{m}{k}\left[\frac{\exp(ky)}{s+k} + \frac{\exp(-sy)}{s-k} - \frac{\exp(-sy)}{s+k}\right] + \frac{m}{k}\left(\frac{\exp(ky)}{s-k}\right)\sum_{n=1}^{\infty} C_n\left(-\frac{V_0}{E}\right)^n \\ &= -\frac{m}{k}\left[\frac{\exp(ky)}{s+k} + \frac{\exp(-sy)}{s-k} - \frac{\exp(-sy)}{s+k}\right] \end{aligned}$$

$$-\frac{m}{k}\left(\frac{\exp(ky)}{s-k}\right)\sum_{n=1}^{\infty}\left(-\frac{V_0}{E}\right)^n\frac{1}{n!}\frac{\partial^n}{\partial u^n}\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)_{u=0}$$
$$=-\frac{m}{k}\left[\frac{\exp(ky)}{s+k}+\frac{\exp(-sy)}{s-k}-\frac{\exp(-sy)}{s+k}\right]-\frac{m}{k}\left(\frac{\exp(ky)}{s-k}\right)\left(\frac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right). \tag{30}$$

It is clear that the sum (from $n = 1$ to $\infty$) is Taylor's series (with a zero remainder) of $(1-\sqrt{1-u})/(1+\sqrt{1-u})$ at $u = -V_0/E$.

Let us now return to the solution of our problem, that is to say to the Green function of the step potential. To do this, we invert the two-sided Laplace transform ($y < 0$ and $x < 0$). The integral must be done on the semi-circles of the right or the left respecting the sign of $(x - y)$ or $x$:

$$g_{-,-}(x,y)=\frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\exp(sx)g^{+}(s)\,\mathrm{d}s$$
$$=\left\{\begin{array}{l}\dfrac{m}{k}\exp(-k(x-y))+\dfrac{m}{k}\exp(k(x+y))\left(\dfrac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right);\ x>y\\ \dfrac{m}{k}\exp(k(x-y))+\dfrac{m}{k}\exp(k(x+y))\left(\dfrac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right);\ x<y\end{array}\right\} \tag{31}$$
$$g_{-,-}(x,y)=\frac{m}{k}\exp(-k|x-y|)+\frac{m}{k}\exp(k(x+y))\left(\frac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right),$$

where we use the notation signs, indexing $g(x, y)$, to indicate the range of $x$ and $y$ respectively. For the case $x > 0$, we must compute $g_n^-(s, y)$ and invert it with respect to the two-sided Laplace transform

$$g_n^-(s,y)=g_n^+(s,y)+\frac{2mV_0}{s^2-k^2}g_{n-1}^-(s,y) \tag{32}$$

$$g_n^-(s,y)=\alpha_n\frac{\exp(ky)}{s-k}+\frac{2mV_0}{s^2-k^2}g_{n-1}^-(s,y), \tag{33}$$

where

$$\alpha_n=-\frac{m}{k}\frac{\left(-\frac{V_0}{E}\right)^n}{n!}\frac{\partial^n}{\partial u^n}\left(\frac{1-\sqrt{1-u}}{1+\sqrt{1-u}}\right)_{u=0}\equiv-\frac{m}{k}\beta_n(k). \tag{34}$$

Step by step we arrive at (for $n \geqslant 1$)

$$g_n^-(s,y)=\alpha_n\frac{\exp(ky)}{s-k}+\frac{2mV_0}{s^2-k^2}g_{n-1}^-(s)$$
$$=\alpha_n\frac{\exp(ky)}{s-k}+\frac{2mV_0}{s^2-k^2}\left(\alpha_{n-1}\frac{\exp(ky)}{s-k}+\frac{2mV_0}{s^2-k^2}g_{n-2}^-(s)\right)$$
$$=\alpha_n\frac{\exp(ky)}{s-k}+\alpha_{n-1}\frac{2mV_0}{(s+k)(s-k)^2}\exp(ky)+\frac{(2mV_0)^2}{(s^2-k^2)^2}g_{n-2}^-(s)$$
$$=\alpha_n\frac{\exp(ky)}{s-k}+\alpha_{n-1}\frac{2mV_0}{(s+k)(s-k)^2}\exp(ky)$$
$$+\frac{(2mV_0)^2}{(s^2-k^2)^2}\left(\alpha_{n-2}\frac{\exp(ky)}{s-k}+\frac{2mV_0}{s^2-k^2}g_{n-3}^-(s)\right)$$
$$=\alpha_n\frac{\exp(ky)}{s-k}+\alpha_{n-1}\frac{2mV_0}{(s+k)(s-k)^2}\exp(ky)+\alpha_{n-2}\frac{(2mV_0)^2}{(s+k)^2(s-k)^3}\exp(ky)$$

$$+\,\alpha_{n-3}\frac{(2mV_0)^3}{(s+k)^3(s-k)^4}\exp(ky)+\frac{(2mV_0)^4}{(s+k)^4(s-k)^4}g^-_{n-4}(s)+\cdots$$

$$+\,\alpha_1\frac{(-2mV_0)^{n-1}}{(s+k)^{n-1}(s-k)^n}\exp(ky)+\frac{(-2mV_0)^n}{(s+k)^n(s-k)^n}g^-_0(s)$$

$$=\exp(ky)\left[\sum_{p=1}^{n}\alpha_{n-p+1}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}-\frac{m}{k}\frac{(2mV_0)^n}{(s+k)^{n+1}(s-k)^n}\right]$$

$$g^-_n(s,y)=-\frac{m}{k}\exp(ky)\left[\sum_{p=1}^{n}\beta_{n-p+1}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}+\frac{(2mV_0)^n}{(s+k)^{n+1}(s-k)^n}\right]. \tag{35}$$

Summing up now the series $g^-(s,y)$

$$g^-(s,y)=g^-_0(s,y)+\sum_{n=1}^{\infty}g^-_n(s,y)=g^-_0(s,y)-\frac{m}{k}\exp(ky)$$

$$\times\sum_{n=1}^{\infty}\left[\sum_{p=1}^{n}\beta_{n-p+1}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}+\frac{(2mV_0)^n}{(s+k)^{n+1}(s-k)^n}\right]$$

$$g^-(s,y)=-\frac{m}{k}\exp(ky)\sum_{n=1}^{\infty}\sum_{p=1}^{n}\beta_{n-p+1}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}$$

$$-\frac{m}{k}\exp(ky)\sum_{n=0}^{\infty}\frac{(2mV_0)^n}{(s+k)^{n+1}(s-k)^n}. \tag{36}$$

$g^-_0(s,y)$ is incorporated in the last sum (by starting the sum by zero). The double sum which has a convolution structure can be written as

$$\sum_{n=1}^{\infty}\sum_{p=1}^{n}\beta_{n-p+1}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}=\left(\sum_{n=1}^{\infty}\beta_n\right)\left(\sum_{p=1}^{\infty}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}\right) \tag{37}$$

then

$$g^-(s,y)=-\frac{m}{k}\exp(ky)*\left[\left(\sum_{n=1}^{\infty}\beta_n\right)\left(\sum_{p=1}^{\infty}\frac{(2mV_0)^{p-1}}{(s+k)^{p-1}(s-k)^p}\right)+\sum_{n=0}^{\infty}\frac{(2mV_0)^n}{(s+k)^{n+1}(s-k)^n}\right]$$

$$g^-(s,y)=-\frac{m}{k}\exp(ky)*\left[\left(\sum_{n=1}^{\infty}\beta_n\right)\left(\frac{1}{(s-k)}\sum_{p=0}^{\infty}\frac{(2mV_0)^p}{(s+k)^p(s-k)^p}\right)\right.$$

$$\left.+\frac{1}{(s+k)}\sum_{n=0}^{\infty}\frac{(2mV_0)^n}{(s+k)^n(s-k)^n}\right]. \tag{38}$$

Summing $\beta_n$ gives $(1-\sqrt{1+V_0/E})/(1+\sqrt{1+V_0/E})$ whereas the second and the third sums are geometrical one, that is,

$$g^-(s,y)=-\frac{m}{k}\exp(ky)*\left[\frac{(1-\sqrt{1+V_0/E})}{(1+\sqrt{1+V_0/E})}\frac{1}{(s-k)}\frac{1}{1-\frac{(2mV_0)}{(s^2-k^2)}}+\frac{1}{(s+k)}\frac{1}{1-\frac{(2mV_0)}{(s^2-k^2)}}\right]$$

$$g^-(s,y)=-\frac{m}{k}\exp(ky)*\left[\frac{(1-\sqrt{1+V_0/E})}{(1+\sqrt{1+V_0/E})}\frac{(s+k)}{(s^2-(k^2+2mV_0))}+\frac{(s-k)}{(s^2-(k^2+2mV_0))}\right]. \tag{39}$$

Let us now invert the last expression (for $x > 0$) taking the residue at the pole $s = -\mu = -\sqrt{k^2 + 2mV_0}$:

$$
\begin{aligned}
g_{+,-}(x, y) &= \frac{1}{2\mathrm{i}\pi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} g^{-}(s) \exp(sx)\,\mathrm{d}s \\
g_{+,-}(x, y) &= -\frac{m}{k} \exp(ky - \mu x) \left[ \frac{k-\mu}{k+\mu} \frac{-\mu+k}{-2\mu} + \frac{\mu+k}{2\mu} \right] = \frac{-2m}{k+\mu} \exp(ky - \mu x).
\end{aligned}
\tag{40}
$$

The result for $y > 0$ and $x < 0$ can be immediately written because the end points $x$ and $y$ play symmetrical roles:

$$
g_{-,+}(x, y) = -\frac{m}{k} \exp(ky - \mu x) \left[ \frac{k-\mu}{k+\mu} \frac{-\mu+k}{-2\mu} + \frac{\mu+k}{2\mu} \right] = \frac{-2m}{k+\mu} \exp(kx - \mu y). \tag{41}
$$

Note at this point, that in [8], there is a mistake because the author equalizes $g_{+,-}(x, y)$ with $g_{-,+}(x, y)$ whereas the result of [10] is identical to our calculation.

## 4. Region $y > 0$

As above, we must write, for the case $y > 0$, the initial values of $g_0^-(s, y)$ and $g_0^+(s, y)$:

$$
\begin{aligned}
g_0^-(s, y) &= -\frac{m}{k} \int_0^{\infty} \exp(-k|x-y|) \exp(-sx)\,\mathrm{d}x \\
&= -\frac{m}{k} \int_0^{y} \exp(k(x-y)) \exp(-sx)\,\mathrm{d}x - \frac{m}{k} \int_y^{\infty} \exp(k(x-y)) \exp(-sx)\,\mathrm{d}x \\
g_0^-(s, y) &= \frac{-m}{k(s-k)} (\exp(-ky) - \exp(-sy)) - \frac{m \exp(-sy)}{k(s+k)}
\end{aligned}
\tag{42}
$$

$$
g_0^+(s, y) = \frac{m}{k} \int_{-\infty}^{0} \exp(-k|x-y|) \exp(-sx)\,\mathrm{d}x = -\frac{m}{k} \frac{\exp(-ky)}{(s-k)}, \tag{43}
$$

then

$$
g_1^+(s, y) - g_1^-(s, y) = \frac{-2mV_0}{(s^2 - k^2)} g_0^-(s, y) \equiv h_1(s, y). \tag{44}
$$

Using the non-homogeneous problem of Hilbert (with $\varphi_0^-(s) = \varphi_0^+(s) = 1$ as above and also $P(s) = 0$) as in [12], we find that

$$
\begin{aligned}
g_1^+(s, y) &= \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{h_1(u, y)\,\mathrm{d}u}{u - s} = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}u \left( \frac{1}{u-s} \right) \frac{-2mV_0}{(u^2 - k^2)} g_0^-(u, y); \qquad s < 0 \\
&= \frac{m}{k} 2mV_0 \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}u \frac{1}{(u-s)} * \left( \frac{\exp(-ky)}{(u+k)(u-k)^2} \right. \\
&\quad \left. - \frac{\exp(-uy)}{(u+k)(u-k)^2} + \frac{\exp(-uy)}{(u-k)(u+k)^2} \right) \\
&= -\frac{m}{k} 2mV_0 \left( \frac{y}{2k} + \frac{1}{(2k)^2} \right) \frac{\exp(-ky)}{s-k} \qquad \text{the pole is at} \quad u = k
\end{aligned}
\tag{45}
$$

$$
g_1^-(s, y) = g_1^+(s, y) - h_1(s, y) = -\frac{m}{k} 2mV_0 \left( \frac{y}{2k} + \frac{1}{(2k)^2} \right) \frac{\exp(-ky)}{s-k} + \frac{2mV_0}{(s^2 - k^2)} g_0^-(s, y). \tag{46}
$$

Let us now look at the second term of the series

$$g_2^+(s,y) - g_2^-(s,y) = \frac{-2mV_0}{(s^2-k^2)} g_1^-(s,y) \equiv h_2(s,y) \tag{47}$$

as above we can write

$$\begin{aligned} g_2^+(s,y) &= \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{h_2(u,y)\,\mathrm{d}u}{u-s} = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}u}{u-s}\left(\frac{-2mV_0}{(u^2-k^2)} g_1^-(u,y)\right) \\ &= -\mathrm{residue}(u=k) = (-2mV_0)^2 \left[\frac{-1}{(2k)^2}\frac{\exp(-ky)}{s+k}\left(\frac{y}{2k}+\frac{1}{(2k)^2}\right)\right. \\ &\quad + \frac{\exp(-ky)}{s+k}\left(\frac{y}{2k}+\frac{1}{(2k)^2}\right)\frac{1}{(s-k)^2} + \frac{\exp(-ky)}{s-k}\left(\frac{y^2}{2(2k)^2}+\frac{3y}{(2k)^3}+\frac{3}{(2k)^4}\right) \\ &\quad \left. - \frac{\exp(-ky)}{(s-k)^2}\left(\frac{y}{(2k)^2}+\frac{1}{(2k)^3}\right)\right]. \end{aligned} \tag{48}$$

We note and generalize that there are terms (proportional to $\frac{1}{s-k}$) contributing to the integral

$$g_n(x,y) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} g_n^+(s,y)\exp(sx)\,\mathrm{d}s \tag{49}$$

and other terms whose contribution to the above integral can be reduced in integrals over terms also proportional to $\frac{1}{s-k}$. The rest of terms of $g_n^+(s,y)$ do not contribute to the integral. These remarks lead us to separate $g_n^+(s,y)$ into two parts: one of them say, relevant, and proportional to $\frac{1}{s-k}$, and another irrelevant part not contributing to the integral. Then, write the relevant part of $g_n^+(s,y)$ as

$$\begin{aligned} g_n^+(s,y)_{\mathrm{rel}} &= -\frac{m}{k}\left(\frac{V_0}{E}\right)^n \frac{\exp(-ky)}{s-k}\sum_{p=0}^{n} a_n(p)y^p \\ g^+(s,y) &= \sum_{n=0}^{\infty} g_n^+(s,y) = -\frac{m}{k}\frac{\exp(-ky)}{s-k}\sum_{n=0}^{\infty}\left(\frac{V_0}{E}\right)^n\sum_{p=0}^{n} a_n(p)y^p. \end{aligned} \tag{50}$$

We can get the coefficients $a_n(p)$ in the following manner. Indeed we have (for $y > 0$ and $x < 0$ in the last section)

$$g(x,y) = \frac{1}{2\mathrm{i}\pi}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} g^+(s,y)\exp(sx)\,\mathrm{d}s = -\frac{m}{k}\frac{2k}{k+\mu}\exp(kx-\mu y) \tag{51}$$

$$g(x,y) = -\left(\sum_{n=0}^{\infty}\left(\frac{V_0}{E}\right)^n\sum_{p=0}^{n} a_n(p)y^p\right)\frac{m}{k}\exp(k(x-y)) \tag{52}$$

then

$$\sum_{n=0}^{\infty}\left(\frac{V_0}{E}\right)^n\sum_{p=0}^{n} a_n(p)y^p = \frac{2k\exp(y(k-\mu))}{k+\mu}. \tag{53}$$

Let us now return to the recursive equation

$$g_n^+(s,y) - g_n^-(s,y) = b(s)g_{n-1}^-(s,y), \tag{54}$$

where

$$b(s) = \frac{-2mV_0}{s^2-k^2}. \tag{55}$$

Following step by step we find

$$\begin{aligned} g_n^-(s,y) &= g_n^+(s,y) + (-b(s))g_{n-1}^-(s,y) \\ &= g_n^+(s,y) + (-b(s))\big[g_{n-1}^+(s,y) + (-b(s))g_{n-2}^-(s,y)\big]\cdots \end{aligned} \tag{56}$$

and finally

$$g_n^-(s,y) = \sum_{m=2}^{n}(-b(s))^{n-m} g_m^+(s,y) + (-b(s))^{n-1} g_1^-(s,y). \tag{57}$$

Then the series

$$\begin{aligned} g^-(s,y) &= g_0^-(s,y) + g_1^-(s,y) + \sum_{n=2}^{\infty} g_n^-(s,y) \\ &= g_0^-(s,y) + \frac{g_1^-(s,y)}{1+b(s)} + \frac{\sum_{n=2}^{\infty} g_n^+(s,y)}{1+b(s)} \\ &= g_0^-(s,y) + \frac{g_1^-(s,y)}{1+b(s)} - \frac{g_0^+(s,y)+g_1^+(s,y)}{1+b(s)} + \frac{\sum_{n=0}^{\infty} g_n^+(s,y)}{1+b(s)} \\ &= g_0^-(s,y) + \frac{g_1^-(s,y)}{1+b(s)} - \frac{g_0^+(s,y)+g_1^+(s,y)}{1+b(s)} \\ &\quad - \frac{1}{1+b(s)}\frac{m}{k}\frac{\exp(-ky)}{s-k}\left(\sum_{n=0}^{\infty}\left(\frac{V_0}{E}\right)^n \sum_{p=0}^{n} a_n(p)y^p\right) \\ &= g_0^-(s,y) + \frac{g_1^-(s,y)}{1+b(s)} - \frac{g_0^+(s,y)+g_1^+(s,y)}{1+b(s)} \\ &\quad - \frac{1}{1+b(s)}\frac{m}{k}\frac{\exp(-ky)}{s-k}\left(\frac{2k\exp y(k-\mu)}{k+\mu}\right). \end{aligned}$$

After some simplifications we arrive at

$$g^-(s,y) = -\frac{m}{k}2k\left(\frac{s+k}{s+\mu}\right)\left(\frac{\exp(-\mu y)}{(k+\mu)(\mu-s)} + \frac{\exp(-sy)}{(s+k)(s-\mu)}\right). \tag{58}$$

Now by inverting the last expression of $g^-(s,y)$, we find for $x > y > 0$

$$\begin{aligned} g_{+,+}(x,y) &= \frac{1}{2i\pi}\int_{-i\infty}^{+i\infty} g^-(s,y)\exp(sx)\,\mathrm{d}s \\ &= -2m\exp(-\mu y)\frac{(k-\mu)\exp(-\mu x)}{2\mu(k+\mu)} - 2m\frac{\exp(-\mu(x-y))}{-2\mu} \\ &= -\frac{m}{\mu}\left(\frac{k-\mu}{k+\mu}\exp(-\mu(x+y)) - \exp(-\mu(x-y))\right) \end{aligned} \tag{59}$$

which agrees with the results of [8] and [10].

At the end we summarize our main results (31), (40), (41) and (59) in a compact form as formula (8) in [10]:

$$\begin{aligned} G(x,y;E) &= \theta(-x)\theta(-y)\frac{m}{k}\left(\exp(-k|x-y|) + \exp(k(x+y))\left(\frac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right)\right) \\ &\quad + \theta(x)\theta(y)\frac{m}{\mu}\left(\exp(-\mu(x-y)) - \frac{k-\mu}{k+\mu}\exp(-\mu(x+y))\right) \\ &\quad + \theta(-x_<)\theta(x_>)\left(\frac{-2m}{k+\mu}\exp(kx_< - \mu x_>)\right). \end{aligned} \tag{60}$$

## 5. Conclusion

We have calculated the Green function for the step potential, following the original framework of the perturbation theory in the path integral formalism proposed earlier by Feynman. We estimate that this work is original since none of the papers found in the literature are treating this problem by summing exactly the perturbation series. We intend in the future to extend this method to other problems with boundary conditions as barriers and multi-scaled potential.

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